foot of the altitude of $\triangle ABM$ from M and let $A - M_1 - B$. Prove that then $\overline{MA} > \overline{MB}$ if and only if $\overline{M_1A} > \overline{M_1B}$.

8. If M is the midpoint of $\overline{BC}$ then $\overline{AM}$ is called a **median** of $\triangle ABC$. Consider $\triangle ABC$ such that $\overline{AB} < \overline{AC}$. Let E, D and H denote the points in which bisector of angle, median and altitude from A intersect line $\overleftrightarrow{BC}$, respectively. Show that (a) $\measuredangle AEB < \measuredangle AEC$; (b) $\overline{BE} < \overline{CE}$; (c) we have $H - E - D$.

9. (a.) Prove that in a neutral geometry if $\triangle ABC$ is isosceles with base $\overline{BC}$ then the following are collinear: (i) the median from A; (ii) the bisector of $\measuredangle A$; (iii) the altitude from A; (iv) the perpendicular bisector of $\overline{BC}$. (b.) Conversely, in a neutral geometry prove that if any two of (i)-(iv) are collinear then the triangle is isosceles (six different cases).

10. Show that the conclusion of the Pythagorean Theorem is not valid in the Poincaré Plane by considering $\triangle ABC$ with $A(2,1)$, $B(0,\sqrt{5})$, and $C(0,1)$. Thus the Pythagorean Theorem does not hold in every neutral geometry.

Theorem In a neutral geometry, if $\overrightarrow{BD}$ is the bisector of $\measuredangle ABC$ and if E and F are the feet of the perpendiculars from D to $\overleftrightarrow{BA}$ and $\overleftrightarrow{BC}$ then $\overline{DE} \cong \overline{DF}$.

11. Prove the above Theorem. [Th 6.4.7, p 148]

20 Circles and Their Tangent Lines

Definition. **(circle with center C and radius r, chord, diameter, radius segment).** If C is a point in a metric geometry $(\mathcal{S}, \mathcal{L}, d)$ and if $r > 0$, then

$$\mathcal{C} = \mathcal{C}_r(C) = \{P \in \mathcal{S} \mid PC = r\}$$

is a circle with center C and radius r. If A and B are distinct points of $\mathcal{C}$ then $\overline{AB}$ is a chord of $\mathcal{C}$. If the center C is a point on the chord $\overline{AB}$, then $\overline{AB}$ is a diameter of $\mathcal{C}$. For any $Q \in \mathcal{C}$, $\overline{CQ}$ is called a radius segment of $\mathcal{C}$.

1. Find and sketch the circle of radius 1 with center $(0,0)$ in the Euclidean Plane and in the Taxicab Plane. [Ex 6.5.1, p150]

2. Consider $\{\mathbb{R}^2, \mathcal{L}_E\}$ with the max distance d_s (recall $d_s(P,Q) = \max\{|x_1 - x_2|, |y_1 - y_2|\}$ where $P(x_1, y_1)$ and $Q(x_2, y_2)$ denote two points in $\mathbb{R}^2$). Sketch the circle $\mathcal{C}_1((0,0))$.

3. Show that $\mathcal{A} = \{(x,y) \in \mathbb{H} \mid x^2 + (y-5)^2 = 16\}$ is the Poincaré circle $\mathcal{C}$ with center $(0,3)$ and radius $\ln 3$. [Ex 6.5.2, p151]

Our first result tells us that in a neutral geometry the center and radius of a circle are determined by any three points on the circle.

Theorem. In a neutral geometry, let $\mathcal{C}_1 = \mathcal{C}_r(C)$ and $\mathcal{C}_2 = \mathcal{C}_s(D)$. If $\mathcal{C}_1 \cap \mathcal{C}_2$ contains at least three points, then $C = D$ and $r = s$. Thus, three points of a circle in a neutral geometry uniquely determine that circle.

4. Prove the above Theorem. [Th 6.5.3, p152]

Corollary. For any circle in a neutral geometry, the perpendicular bisector of any chord contains the center.

5. If $\overline{AB}$ is a chord of a circle in a neutral geometry but is not a diameter, prove that the line through the midpoint of $\overline{AB}$ and the center of the circle is perpendicular to $\overline{AB}$.

6. Prove that a line in a neutral geometry intersects a circle at most twice.

Definition. **(interior, exterior).** Let $\mathcal{C}$ be the circle with center C and radius r. The interior of $\mathcal{C}$ is the set $\mathrm{int}(\mathcal{C}) = \{P \in \mathcal{S} \mid CP < r\}$. The exterior of $\mathcal{C}$ is the set $\mathrm{ext}(\mathcal{C}) = \{P \in \mathcal{S} \mid CP > r\}$.

Theorem. If $\mathcal{C}$ is a circle in a neutral geometry then $\mathrm{int}(\mathcal{C})$ is convex.

7. Prove the above Theorem. [Th 6.5.5, p153]

Definition. **(tangent, point of tangency).** In a metric geometry, a line ℓ is a tangent to the circle $\mathcal{C}$ if $\ell \cap \mathcal{C}$ contains exactly one point (which is called the point of tangency). ℓ is called a secant of the circle $\mathcal{C}$ if $\ell \cap \mathcal{C}$ has exactly two points.

8. In the Taxicab Plane prove that for the circle $\mathcal{C} = \mathcal{C}_1((0,0))$: (a). There are exactly four points at which a tangent to $\mathcal{C}$ exists. (b). At each point in part (a) there are infinitely many tangent lines.

Theorem. In a neutral geometry, let $\mathcal{C}$ be a circle with center C and let $Q \in \mathcal{C}$. If t is a line through Q, then t is tangent to $\mathcal{C}$ if and only if t is perpendicular to the radius segment $\overline{CQ}$.

9. Prove the above Theorem. [Th 6.5.6, p154]

Corollary. **(Existence and Uniqueness of Tangents).** In a neutral geometry, if $\mathcal{C}$ is a circle and $Q \in \mathcal{C}$ then there is a unique line t which is tangent to $\mathcal{C}$ and whose point of tangency is Q.

10. Prove the above Corollary. [Cor 6.5.7, p155]

Definition. **(continuous).** Function $h : \mathbb{R} \to \mathbb{R}$ is continuous at $t_0 \in \mathbb{R}$ if for every $\varepsilon > 0$ there is a $\delta > 0$ such that $|h(t) - h(t_0)| \leq \varepsilon$ if $|t - t_0| < \delta$. (Thus if t is "near" t_0 then $h(t)$ is "near" $h(t_0)$).

Intermediate Value Theorem. If $h : [a,b] \to \mathbb{R}$ is continuous at every $t_0 \in [a,b]$ and if y is a number between $h(a)$ and $h(b)$ then there is a point $s \in [a,b]$ with $h(s) = y$.

Theorem. Let r be a positive real number and let A, B, C be points in a neutral geometry such that $AC < r$ and $\overrightarrow{AB} \perp \overline{AC}$. Then there is a point $D \in \overrightarrow{AB}$ with $CD = r$.

11. Prove the above Theorem. [Th 6.5.8, p156]

Theorem. **(Line-Circle Theorem).** In a neutral geometry, if a line ℓ intersects the interior of a circle $\mathcal{C}$, then ℓ is a secant.

12. Prove the above Theorem. [Th 6.5.9, p157]

Theorem. **(External Tangent Theorem).** In a neutral geometry, if $\mathcal{C}$ is a circle and $P \in \text{ext}(\mathcal{C})$, then there are exactly two lines through P tangent to $\mathcal{C}$.

13. Prove the above Theorem. [Th 6.5.10, p158]

14. In a neutral geometry, if $\mathcal{C}$ is a circle with $A \in \text{int}(\mathcal{C})$ and $B \in \text{ext}(\mathcal{C})$, prove that $\overline{AB} \cap \mathcal{C} \neq \emptyset$.

21 The Two Circle Theorem

From previus lesson we know that two distinct circles in a neutral geometry intersect in at most two points. The main point of this section is to give a condition for when two circles intersect in exactly two points. This result, called the Two Circle Theorem, will follow directly from a converse of the Triangle Inequality.

Theorem. **(Sloping Ladder Theorem).** In a neutral geometry with right triangles $\triangle ABC$ and $\triangle DEF$ whose right angles are at C and F, if $\overline{AB} \cong \overline{DE}$ and $\overline{AC} > \overline{DF}$, then $\overline{BC} < \overline{EF}$.

1. Prove the above Theorem. [Th 6.6.1, p160]

Theorem. Let $\overline{AB}$ and $\overline{DE}$ be two chords of the circle $\mathcal{C} = \mathcal{C}_r(C)$ in a neutral geometry. If $\overline{AB}$ and $\overline{DE}$ are both perpendicular to a diameter of $\mathcal{C}$ at points P and Q with $C - P - Q$, then $DQ < AP < r$.

2. Prove the above Theorem.

Theorem. **(Triangle Construction Theorem).** Let $\{\mathcal{S}, \mathcal{L}, d, m\}$ be a neutral qeometry and let a, b, c be three positive numbers such that the sum of any two is greater than the third. Then there is a triangle in $\mathcal{S}$ whose sides have length a, b and c.

3. Prove the above Theorem. [Th 6.6.3, p161]

Theorem. **(Two Circle Theorem).** In a neutral geometry, if $\mathcal{C}_1 = \mathcal{C}_b(A)$, $\mathcal{C}_2 = \mathcal{C}_a(B)$, $AB = c$, and if each of a, b, c is less than the sum of the other two, then $\mathcal{C}_1$ and $\mathcal{C}_2$ intersect in exactly two points, and these points are on opposite sides of $\overleftrightarrow{AB}$.

4. Prove the above Theorem.

Theorem. If a protractor geometry satisfies SSS and both the Triangle Inequality and the Two Circle Theorem with the neutral hypothesis omitted, then it also satisfies SAS and is a neutral geometry.

5. Prove the above Theorem. [Th 6.6.6, p164]

6. Prove that in a neutral geometry, two circles $\mathcal{C}_1$ and $\mathcal{C}_2$intersect in exactly two points if and only $\mathcal{C}_1 \cap \text{int}(\mathcal{C}_2) \neq \emptyset$ and $\mathcal{C}_1 \cap \text{ext}(\mathcal{C}_2) \neq \emptyset$.

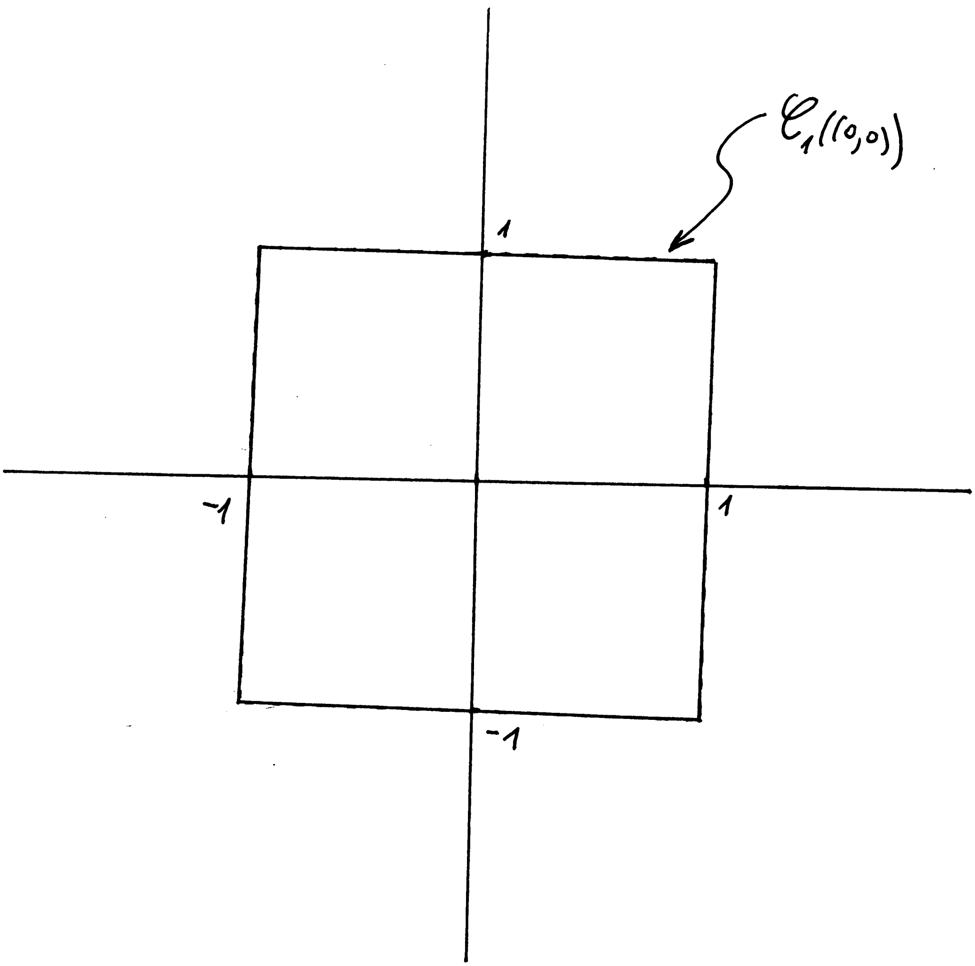
7. Prove that in a neutral geometry a circle of radius r has a chord of length c if and only if $0 < c \leq 2r$.

8. In a neutral geometry prove that for any $s > 0$ there is an equilateral triangle each of whose sides has length s.

(#) Posmatrajmo $\{\mathbb{R}^2, \mathcal{L}_E\}$ sa maksimalnom udaljenošću d_s ($d_s(P,Q) = \max\{|x_1-x_2|, |y_1-y_2|\}$ gdje je $P(x_1,y_1)$, $Q(x_2,y_2)$). Skicirati krug $\mathcal{C}_1((0,0))$.

Rj.

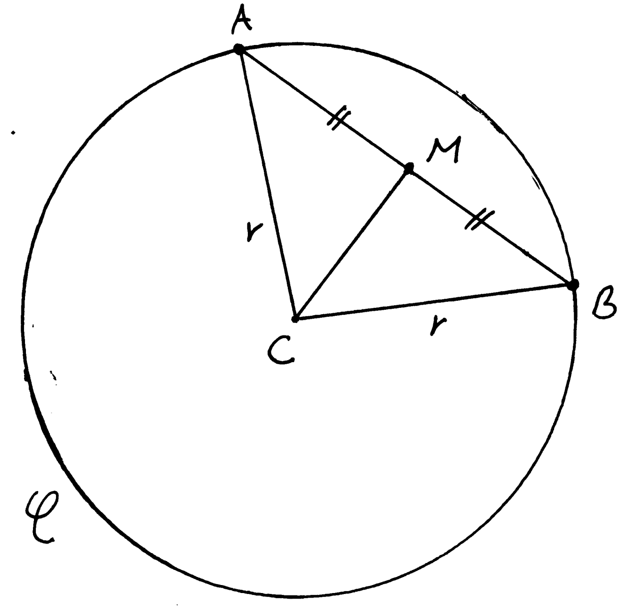
$$\mathcal{C}_1((0,0)) \stackrel{\text{def.}}{=} \{P \in \mathcal{S} \mid d_s(P,(0,0)) = 1\} = \{(x,y) \in \mathcal{S} \mid \max\{|x|,|y|\} = 1\}$$



(#) Neka je $\overline{AB}$ tetiva kruga u neutralnoj geometriji koja nije dijametar. Pokazati da je prava koja prolazi kroz sredinu tetive $\overline{AB}$ i centar kruga okomita na $\overline{AB}$.

Rj: Označimo sa $\mathcal{C} = \mathcal{C}_r(C)$ dati krug i neka je $\overline{AB}$ data tetiva. Označimo sa M sredinu tetive $\overline{AB}$ i posmatrajmo trouglove $\triangle AMC$ i $\triangle BMC$.

$(A, B \in \mathcal{C} \Rightarrow AC = r,\ BC = r)$



$$\left.\begin{array}{l} \overline{AC} \cong \overline{BC} \\ \overline{AM} \cong \overline{BM} \\ \overline{CM} \cong \overline{CM} \end{array}\right\} \overset{SSS}{\Longrightarrow} \triangle AMC \cong \triangle BMC$$

$$\Downarrow$$

$$\measuredangle AMC \cong \measuredangle BMC$$

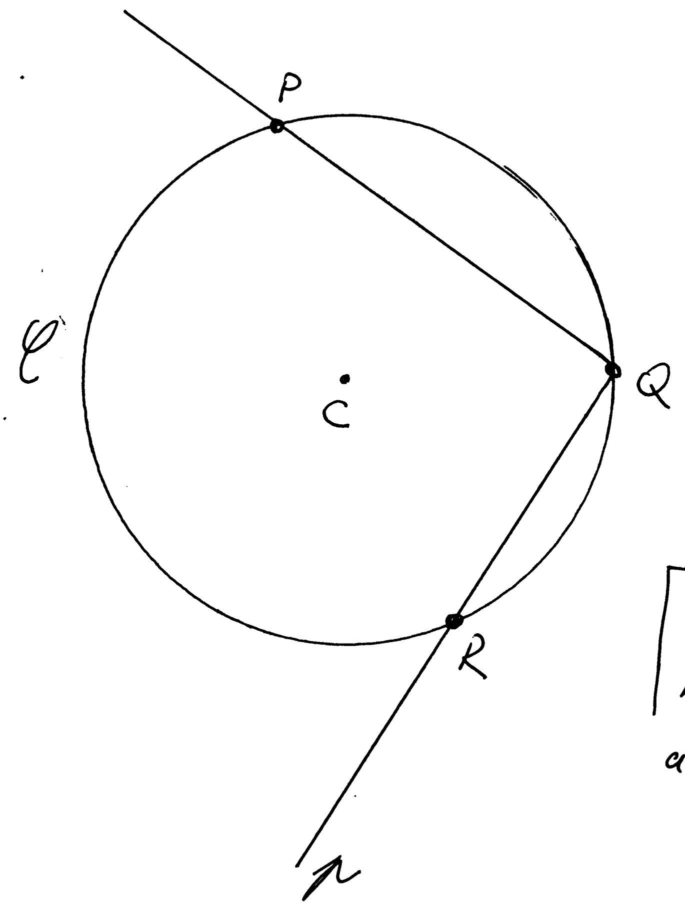
S obzirom da su ova dva suplementarna ugla možemo zaključiti da je $m(\measuredangle AMC) = 90 = m(\measuredangle BMC)$.

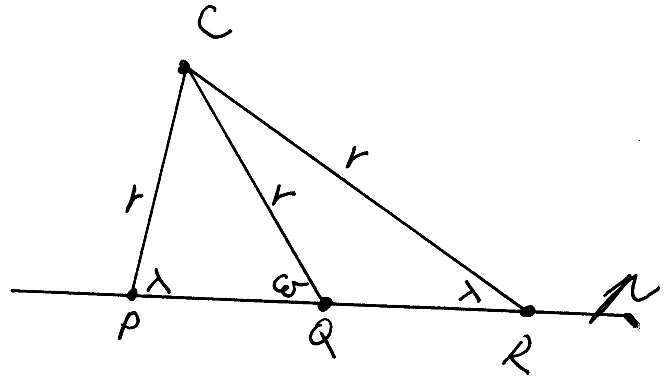
Drugim riječima $\overline{MC} \perp AB \Rightarrow \overleftrightarrow{MC} \perp AB$

q.e.d.

(#) Dokazati da prava u neutralnoj geometriji može da siječe krug u najviše dvije tačke.

Rj: Neka je dat krug $\mathscr{C} = \mathscr{C}_r(C)$ i prava p i pretpostavimo suprotno tvrdnji tj. pretpostavimo da prava p siječe krug u najmanje tri tačke P, Q i R (pretpostavimo i da je $P-Q-R$).





Prisjetimo se jedne od prethodnih teorema koja kaže da ako imamo $\triangle ABC$ i ako je $A-D-C$ tada je $\overline{BD} < \max\{BA, BC\}$

Posmatrajmo $\triangle CPR$. S obzirom da je $PC = RC = r$ to je $\overline{PC} \cong \overline{RC} \Rightarrow \sphericalangle CPR \cong \sphericalangle CRP$. Kako je $\sphericalangle CQP$ vanjski ugao $\triangle CQR$ to je $\sphericalangle CQP > \sphericalangle CRQ$ ($\omega > \lambda$ sa slike)

Ali sad u $\triangle CPQ$ imamo da je $\sphericalangle CQP > \sphericalangle CPQ \Rightarrow$

$\Rightarrow \overline{CP} > \overline{CQ}$

kontradikcija
($CP = r = CQ$)

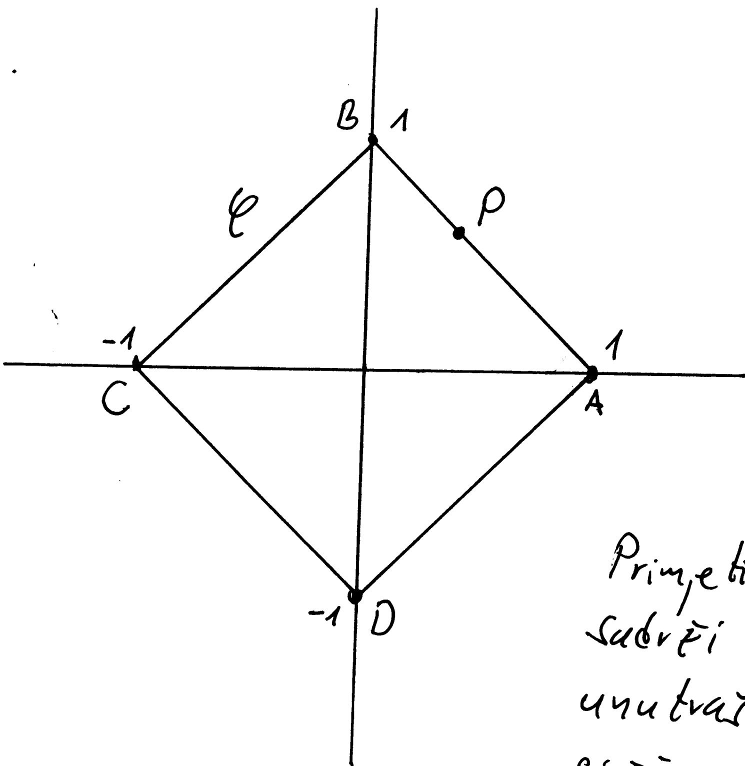
Pretpostavka suprotna tvrdnji nas vodi u kontradikciju pa nije tačna. Prema tome prava može sjeći krug u nula, jednoj ili dvije tačke.

#) U taksi ravni dokazati da za krug $\mathcal{C} = \mathcal{C}_1((0,0))$

(a.) Postoje tačno četiri tačke u kojima možemo povući tangentu na krug $\mathcal{C}$

(b.) U svakoj tački iz (a.) postoji beskonačno mnogo tangentnih linija.

Rj. U taksi ravni $\mathcal{C}_1((0,0)) = \{(x,y) \in \mathbb{R}^2 \mid |x|+|y| = 1\}$



Označimo sa A, B, C i D redom tačke $A(1,0)$, $B(0,1)$, $C(-1,0)$, $D(0,-1)$.

Neka je $P \in \overline{AB}$ t.d. $A-P-B$. Ako je $P(x_1,y_1)$ primijetimo da je tada $0<x_1<1$ i $0<y_1<1$.

Primjetimo da ako je p prava koja sadrži tačku P i neku tačku iz unutrašnjosti kruga $\mathcal{C}$ tada p siječe krug u dvije tačke (ovo se može dokazati korištenjem Crossbar teoreme na trouglove ΔABC i ΔDAC)

(a) Neka je $P \in \overline{AB}$ t.d. $A-P-B$ i pokažimo da ne postoji tangenta u tački P. Ovo ćemo pokazati tako što ćemo posmatrati sve moguće prave kroz tačku P i vidjeti da ni jedna od njih ne siječe krug samo u tački P nego da uvijek postoji najmanje još jedna tačka. Neka je $P(x_1,y_1)$.

Ako je $P \in L_{x_1}$ tada je $Q(x_1,-y_1) \in L_{x_1}$ i imamo da $Q \in L_{x_1}$ i $Q \in \mathcal{C}_1((0,0))$. Drugim riječima L_{x_1} nije tangenta na krug $\mathcal{C}$.

$\Rightarrow$ Ne postoji vertikalna prava koja sadrži P i koja je tangenta na $\mathcal{C}$.

Posmatrajmo sad nevertikalne prave (koje sadrže tačku P. $y = kx+n$)

1° $k \geq 0$ i označimo vertikalnu pravu sa $L_{k,n}$ ($L_{k,n}$ je $y=kx+n$)

(a) $-1 \leq n \leq y_1$

$Q(0,n) \in L_{k,n}$ i $Q \in int(\mathcal{C}_1(1(0,0))$ (ili u slučaju za $n=-1$ $Q \equiv D$).

Kako $L_{k,n}$ sadrži tačku P i tačku Q iz unutrašnjosti to $L_{k,n}$ siječe krug u najmanje još jednoj tački $\Rightarrow$ $L_{k,n}$ nije tangenta

(b) $n > y_1 \Rightarrow y_1 = kx_1 + n \overset{k\geq 0}{\Rightarrow} y_1 =$ neki nenegativan broj + broj koji je veći od y_1 #kontradikcija

Slučaj u kojem je $n \geq y_1$ nije moguć

(c) $n < -1 \Rightarrow y_1 = kx_1 + n \Rightarrow k = \frac{y_1 - n}{x_1}$ (a kako je $-n > 0$, $0 < x_1 < 1$ to je $k > 0$ i $k > n$)

$Q(-\frac{n}{k}, 0) \in L_{k,n}$ i $Q \in int(\mathcal{C}_1(0,0))$

Kako $L_{k,n}$ sadrži tačku P i tačku Q iz unutrašnjosti to $L_{k,n}$ siječe krug u najmanje još jednoj tački $\Rightarrow$ $L_{k,n}$ nije tangenta

2° $k < 0$

Slično (ZA VJEŽBU)

Na osnovu 1° i 2° vidimo da ne postoji nevertikalna prava u tački P t.d. ima sa krugom $\mathcal{C}$ samo jednu zajedničku tačku. $\Rightarrow$ Ni u jednoj tački iz $int(\overline{AB})$ ne možemo povući tangentu na krug $\mathcal{C}$. Slično za tačke iz $int(\overline{BC})$, $int(\overline{CD})$ i $int(\overline{DA})$.

Možemo zaključiti da samo u tačkama A, B, C i D možemo povući tangentu na krug $\mathcal{C}$.

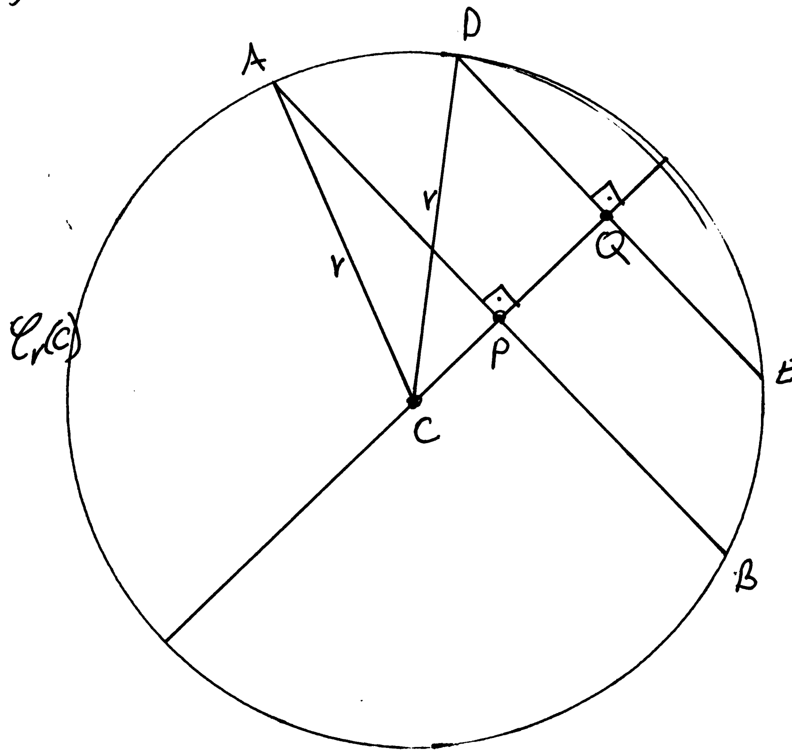
(b) ZA VJEŽBU. (npr. za tačku A $y = -x + n$ $\forall n > 1$ nek...)

Teorem

Neka su $\overline{AB}$ i $\overline{DE}$ dvije tetive kruga $\mathscr{C}=\mathscr{C}_r(C)$ u neutralnoj geometriji. Ako su obe ove tetive $\overline{AB}$ i $\overline{DE}$ okomite na dijametar kruga $\mathscr{C}$ u tačkama P i Q gdje je poredak $C-P-Q$ tada je $DQ < AP < r$.

(#) Dokazati teoremu iznad.

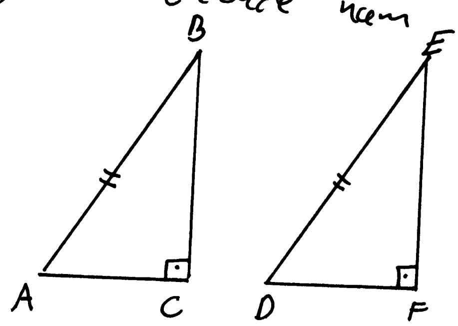
Rj.



Uvedimo oznake kao na slici. Primjetimo da su ΔCQD i ΔCPA pravougli trouglovi sa hipotenuzama CD i CA redom pa je $\overline{CD} > \overline{DQ}$ i $\overline{CA} > AP$.

S obzirom da je $CD = r = CA$ to je $r > DQ$ i $r > AP$ ...(1)

Da bi pokazali da je $AP > DQ$ trebaće nam "Teorem nagnutih ljestvi":



ΔABC
ΔDEF
$\measuredangle C$ prav
$\measuredangle F$ prav
$\overline{AB} \cong \overline{DE}$

$\overline{AC} > \overline{DF} \Rightarrow \overline{BC} < \overline{EF}$

Posmatrajmo trouglove ΔCPA i ΔCQD

$$\left.\begin{array}{l} \measuredangle P, \measuredangle Q \text{ pravi uglovi} \\ \overline{AC} \cong \overline{CD} \\ C-P-Q \Rightarrow \overline{CP} < \overline{CQ} \end{array}\right\} \xrightarrow[\text{ljestvi}]{\text{Teor. nagnutih}} \overline{DQ} < \overline{AP} \quad \ldots(2)$$

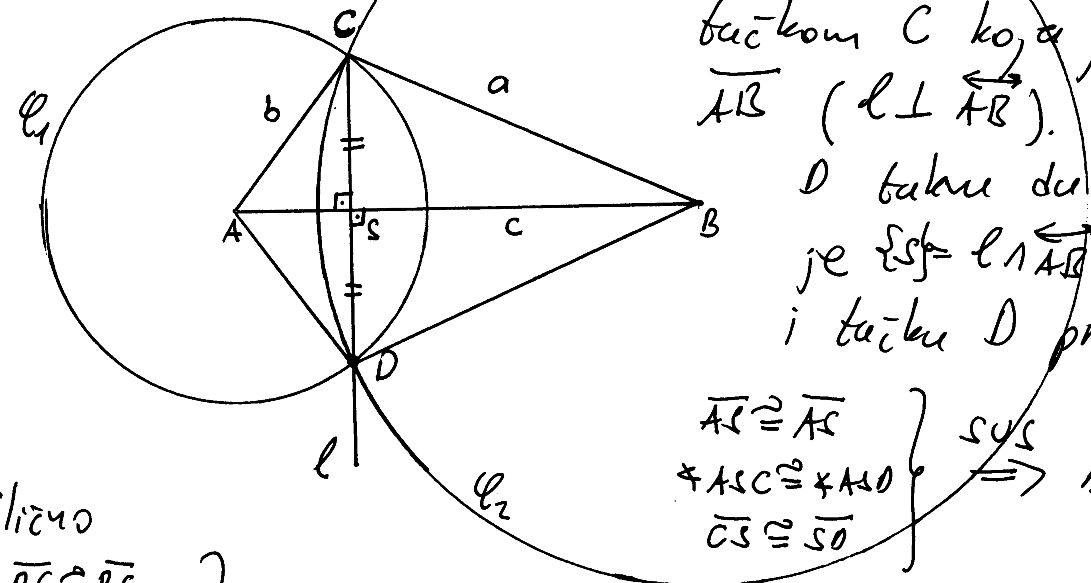
Na osnovu (1) i (2) $DQ < AP < r$ q.e.d.

Teorema (teorema dva kruga)

U neutralnoj geometriji, ako je $\mathcal{C}_1 = \mathcal{C}_b(A)$, $\mathcal{C}_2 = \mathcal{C}_a(B)$, $AB = c$, i ako je svaki od a, b, c manji od sume ostala dva, tada se $\mathcal{C}_1$ i $\mathcal{C}_2$ sijeku u tačno dvije tačke, i ove tačke su sa različitih strana prave $p(A,B) = \overleftrightarrow{AB}$.

(#) Dokazati teoremu iznad.

Rj. Posmatrajmo brojeve a, b i c. Kako su a, b i c pozitivni brojevi i kako je suma bilo koja dva veća od trećeg prema Teoremu konstrukcije trougla postoji ΔABC t.d. $AB = c$, $AC = b$ i $BC = a$. Posmatrajmo krugove $\mathcal{C}_1 = \mathcal{C}_b(A)$ i $\mathcal{C}_2 = \mathcal{C}_a(B)$. Primjetimo da krugovi $\mathcal{C}_1$ i $\mathcal{C}_2$ zadovoljavaju sve uslove iz postavke teoreme. Isto tako primjetimo da kako je $AC = b$ i $BC = a$ to $C \in \mathcal{C}_1 \cap \mathcal{C}_2$. Neka je l poluprava sa početkom tačkom C koja je okomita na $\overleftrightarrow{AB}$ ($l \perp \overleftrightarrow{AB}$). Izaberimo tačku D tako da $\overline{CS} \cong \overline{DS}$, $C-S-D$ gdje je $\{S\} = l \cap \overleftrightarrow{AB}$. Pokažimo da i tačka D pripada presjeku $\mathcal{C}_1 \cap \mathcal{C}_2$.



$$\left.\begin{array}{l} \overline{AS} \cong \overline{AS} \\ \measuredangle ASC \cong \measuredangle ASD \\ \overline{CS} \cong \overline{SD} \end{array}\right\} \overset{SUS}{\Longrightarrow} \Delta ACS \cong \Delta ADS \Downarrow \overline{AC} \cong \overline{AD} \Longrightarrow D \in \mathcal{C}_1 \quad ...(1)$$

Slično

$$\left.\begin{array}{l} \overline{BS} \cong \overline{BS} \\ \measuredangle BSC \cong \measuredangle BSD \\ \overline{CS} \cong \overline{DS} \end{array}\right\} \overset{SUS}{\Longrightarrow} \Delta BCS \cong \Delta BDS \Downarrow \overline{BC} \cong BD \Longrightarrow D \in \mathcal{C}_2 \quad ...(2)$$

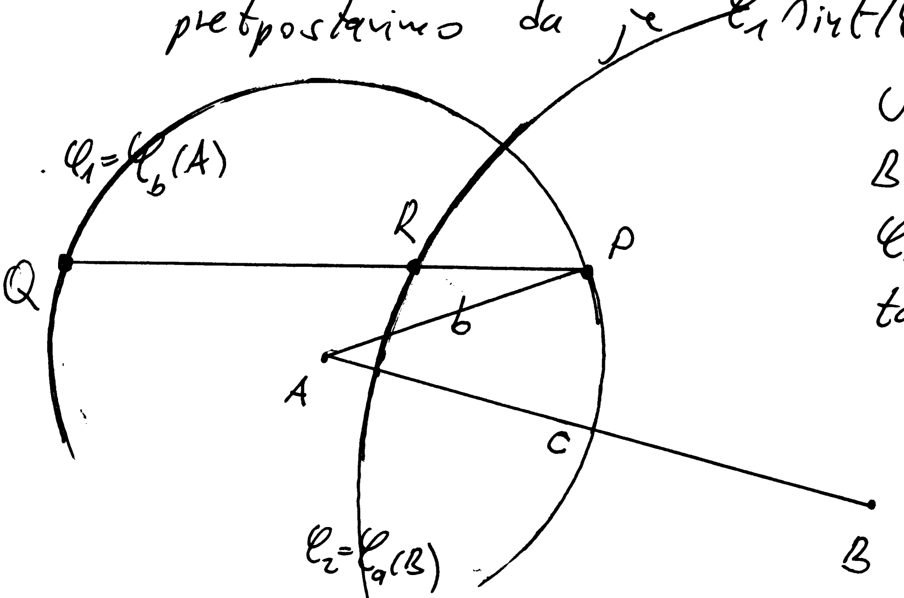
Na osnovu (1) i (2) $\Rightarrow D \in \mathcal{C}_1 \cap \mathcal{C}_2$

Ranije smo pokazali da dva različita kruga ne mogu imati više od dvije različite tačke. Kako je povedeno $C-S-D$ i $S \in \overleftrightarrow{AB}$ to su C i D sa razl. str. pr. $\overleftrightarrow{AB}$.

(#) Dokazati da se u neutralnoj geometriji dva kruga $\mathcal{C}_1$ i $\mathcal{C}_2$ sijeku u tačno dvije tačke ako i samo ako $\mathcal{C}_1 \cap int(\mathcal{C}_2) \neq \emptyset$ i $\mathcal{C}_1 \cap ext(\mathcal{C}_2) \neq \emptyset$.

Rj:

„$\Leftarrow$" Neka su dati krugovi $\mathcal{C}_1 = \mathcal{C}_b(A)$ i $\mathcal{C}_2 = \mathcal{C}_a(B)$ i pretpostavimo da je $\mathcal{C}_1 \cap int(\mathcal{C}_2) \neq \emptyset$ i $\mathcal{C}_1 \cap ext(\mathcal{C}_2) \neq \emptyset$.



Udaljenost između tačaka A i B označimo sa c. Kako je $\mathcal{C}_1 \cap int(\mathcal{C}_2) \neq \emptyset$ to postoji tačka $P \in \mathcal{C}_1 \cap int(\mathcal{C}_2)$.

Za tačku P moguća su dva slučaja

1° $P \in p(A,B) = \overleftrightarrow{AB}$

2° $P \notin \overleftrightarrow{AB}$

Prvi slučaj ostavljamo za vježbu. Posmatrajmo drugi slučaj.

$P \notin p(A,B) = \overleftrightarrow{AB}$ $\Rightarrow$ tačke A, B i P su nekolinearne $\Rightarrow \exists \triangle ABP$.

Prema nejednakosti trougla $BP + AB > AP$ tj. $BP + c > b$.

Kako je $\underbrace{a + c > BP + c}_{(P \in int(\mathcal{C}_2))}$ to je $a + c > b$ ...(1)

S druge strane $AP + BP > AB$ tj. $b + BP > c$.

Kako je $P \in int(\mathcal{C}_2)$ to je $b + a > b + BP$ pa je $b + a > c$ ...(2)

Pokažimo još da je $b + c > a$.

Kako je $\mathcal{C}_1 \cap ext(\mathcal{C}_2) \neq 0$ to postoji tačka $Q \in \mathcal{C}_1 \cap ext(\mathcal{C}_2)$

Kako je $\overline{PQ}$ tetiva kruga $\mathcal{C}_1$ i kako je $P \in int(\mathcal{C}_2)$, $Q \in ext(\mathcal{C}_2)$ to postoji tačka R t.d. $R \in \overline{PQ}$, $P - Q - R$ i $R \in \mathcal{C}_2$. Za tačku R moguća su dva slučaja

1° $R \in p(A,B) = \overleftrightarrow{AB}$

2° $R \notin \overleftrightarrow{AB}$

Prvi slučaj ostavljamo za vježbu. Posmatrajmo drugi slučaj:

$R \notin p(A,B) = \overrightarrow{AB} \Rightarrow A, B, R$ nekolinearne tačke $\Rightarrow \exists \triangle ABR$

Prema nejednakosti trougla $AB + AR > BR$ tj. $c + AR > a$

Kako je $R \in int(\ell_1)$ to je $c + b > c + AR$ pa je $c + b > a$ ...(3)

Sad na osnovu (1), (2) i (3) i teoreme dva kruga, krugovi ℓ_1 i ℓ_2 se sijeku u tačno dvije tačke.

„$\Rightarrow$" Pretpostavimo da se krugovi $\ell_1 = \ell_b(A)$ i $\ell_2 = \ell_a(B)$ sijeku u tačno dvije tačke M i N.

ZAVRŠITI ZA VJEŽBU